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Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups

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Abstract

We study those numerical semigroups that are intersections of symmetric numerical semigroups and we construct an algorithm to find this decomposition. These semigroups are characterized from their pseudo-Frobenius numbers. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A *numerical semigroup* is a subset S of $\mathbb{N}$ closed under addition; it contains the zero and generates $\mathbb{Z}$ as a group (here $\mathbb{N}$ and $\mathbb{Z}$ denote the set of nonnegative integers and the set of integers, respectively). From [2,11] we obtain the following results:

(1) The set $\mathbb{N} \setminus S$ is finite. We refer to the greatest integer not belonging to S as the *Frobenius number* of S and denote it by $g(S)$.

(2) The semigroup S admits a unique minimal system of generators $\{n_1 < \cdots < n_p\}$. The number n_1 is usually called the *multiplicity* of S and p the *embedding dimension* of S, and we denote them by $m(S)$ and $\mu(S)$, respectively.

We say that a numerical semigroup is *symmetric* if for every $z \in \mathbb{Z} \setminus S$, we have that $g(S) - z \in S$. This kind of semigroups are interesting and have been widely studied in

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the literature. In [5], it is shown that S is a symmetric numerical semigroup if and only if $g(S)$ is odd and S is maximal in the set of all numerical semigroups with Frobenius number $g(S)$. Therefore, symmetric semigroups could be an important piece for solving the "Frobenius problem", that is, finding a formula for the Frobenius number of a numerical semigroup in terms of its minimal system of generators (see for instance [13–15]). Numerical semigroups and in particular, those that are symmetric play an important role in ring theory through the concept of the semigroup ring associated to them (see for example [3,4,6,12]). In particular, in [7], it is shown that a numerical semigroup S is symmetric if and only if its semigroup ring $K[S]$ is Gorenstein.

A numerical semigroup is irreducible if it cannot be expressed as an intersection of two numerical semigroups properly containing it. In [10], it is shown that a numerical semigroup S is irreducible if and only if S is maximal in the set of all numerical semigroups with Frobenius number $g(S)$. From [5], we can then deduce that the set of irreducible numerical semigroups with odd (even) Frobenius number coincides with the set of symmetric (pseudo-symmetric) numerical semigroups (see for instance [2] for the definition of pseudo-symmetric numerical semigroup). Hence every numerical semigroup can be expressed as an intersection of numerical semigroups that are either symmetric or pseudo-symmetric. The following natural question then arises: when can a numerical semigroup be expressed as an intersection of only symmetric numerical semigroups? In this paper, we study the semigroups that can be expressed as a finite intersection of symmetric numerical semigroups (ISY-semigroups). Furthermore, we give algorithmic methods to find such a decomposition.

The contents in this paper are organized as follows: In Section 2, we characterize the ISY-semigroups. In Section 3, we introduce the concept of pseudo-Frobenius number of a numerical semigroup and we give a new characterization of ISY-semigroups in terms of their pseudo-Frobenius numbers. In Section 4, we study the class of numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups with the same Frobenius number (ISYG-semigroups) and we give a characterization for them in terms of their pseudo-Frobenius numbers. Such results are used in Section 5 for studying the ISY-semigroups of Type 2. Finally, in Section 6 we deal with the numerical semigroups that can be expressed as a finite intersection of symmetric numerical semigroups with the same multiplicity (ISYM-semigroups).

2. Basic results and a first characterization

An *ISY-semigroup* is a numerical semigroup that can be expressed as a finite intersection of symmetric numerical semigroups. Our aim in this section is to prove Theorem 4 which gives a characterization for this kind of semigroup.

From [5, Proposition 4] we deduce the following result.

Lemma 1. *Let g be an integer number and $\mathscr{S}(g)$ the set of all numerical semigroups with Frobenius number g. Then $S \in \mathscr{S}(g)$ is symmetric if and only if g is odd and S is maximal with respect to set inclusion in $\mathscr{S}(g)$.*

In order to prove Theorem 4 we introduce the following lemmas.

Lemma 2. *If S is a numerical semigroup and x is an odd positive integer not in S, then there exists a symmetric semigroup $\bar{S}$ such that $S \subseteq \bar{S}$ and $g(\bar{S}) = x$.*

Proof. Let $S' = S \cup \{x+1, x+2, \ldots\}$. Clearly, S' is a numerical semigroup and $g(S') = x$. Let $\bar{S}$ be a maximal semigroup in $\mathscr{S}(x)$ such that $S' \subseteq \bar{S}$. By Lemma 1 we can deduce that $\bar{S}$ is symmetric with Frobenius number x and contains S. $\square$

Lemma 3. *Let S be a numerical semigroup and let x be an even positive integer not in S. Then, the following conditions are equivalent:*
(1) *there exists a symmetric semigroup $\bar{S}$ such that $S \subseteq \bar{S}$ and $x \notin \bar{S}$,*
(2) *there exists an odd positive integer y such that $x + y \notin \langle S, y\rangle$.*

Proof. (1) $\Rightarrow$ (2) Let $y = g(\bar{S}) - x$. Since x is even and $g(\bar{S})$ is odd, we have that y is odd (note that, by Lemma 1, the Frobenius number of a symmetric semigroup is always odd). Furthermore, $y = g(\bar{S}) - x \in \bar{S}$, since $x \notin \bar{S}$ and $\bar{S}$ is symmetric. Hence, $\langle S, y\rangle \subseteq \bar{S}$ and thus $x + y = g(\bar{S}) \notin \langle S, y\rangle$.

(2) $\Rightarrow$ (1) Let $S' = \langle S, y\rangle \cup \{x + y + 1, x + y + 2, \ldots\}$. Then S' is a numerical semigroup with odd Frobenius number $x + y$. Using Lemma 2 we deduce that there exists a symmetric semigroup $\bar{S}$ such that $S' \subseteq \bar{S}$ and $g(\bar{S}) = x + y$. Then $S \subseteq \bar{S}$ and $x \notin \bar{S}$, because otherwise, since $y \in \bar{S}$, we would obtain that $g(\bar{S}) = x + y \in \bar{S}$, which is impossible. $\square$

Theorem 4. *Let S be a numerical semigroup. The following conditions are equivalent:*
(1) *S is an ISY-semigroup,*
(2) *for every even positive integer $x \notin S$, there exists an odd positive integer y such that $x + y \notin \langle S, y\rangle$.*

Proof. (1) $\Rightarrow$ (2) Let x be an even positive integer such that $x \notin S$ and let $\bar{S}$ be a symmetric numerical semigroup such that $S \subseteq \bar{S}$ and $x \notin \bar{S}$ (the existence of $\bar{S}$ is guaranteed because S is ISY-semigroup). Applying Lemma 3, we deduce that there exists an odd positive integer y such that $x + y \notin \langle S, y\rangle$.

(2) $\Rightarrow$ (1) If x is an odd positive integer such that $x \notin S$, then let S_x be a symmetric numerical semigroup with $S \subseteq S_x$ and $g(S_x) = x$ (Lemma 2 guarantees the existence of S_x). If x is an even positive integer such that $x \notin S$ then, by Lemma 3, we deduce that there exists a symmetric numerical semigroup S_x fulfilling that $S \subseteq S_x$ and $x \notin S_x$. Finally, it is clear that $S = \bigcap_{x \notin S} S_x$. $\square$

The next result has an immediate proof.

Lemma 5. *Let $S, S_1, \ldots, S_n$ be numerical semigroups such that $S = S_1 \cap \cdots \cap S_n$. Then $g(S) = \max\{g(S_1), \ldots, g(S_n)\}$.*

As a consequence of this lemma and from the fact that the Frobenius number of a symmetric numerical semigroup is always odd, we get the next result.

Lemma 6. *If S is an ISY-semigroup, then $g(S)$ is odd.*

We can see, with the following example, that the converse of this result is not true.

Example 7. If $S=\langle 4,5,6,7\rangle$, then $g(S)=3$. Now we see that S is not an ISY-semigroup and for this we use Theorem 4. In fact, $2\notin S$ (2 is even) and for every odd positive integer y, we have that $2+y\in\langle S,y\rangle$.

Arguing as this example and using Lemma 6, the reader can check the following result.

Proposition 8. *If $m\geqslant 3$, then $S=\langle m,m+1,\dots,m+(m-1)\rangle$ is not an ISY-semigroup.*

Note that if S is a numerical semigroup and $\mu(S)\in\{1,2\}$ then S is symmetric (see for instance [6]). Then, $\langle 5,7\rangle\cap\langle 5,8\rangle=\langle 5,21,24,28,32\rangle$ is an ISY-semigroup.

3. The pseudo-Frobenius numbers of a numerical semigroup

Let S be a numerical semigroup. We say that an element of $x\in\mathbb{Z}$ is a *pseudo-Frobenius number* of S if $x\notin S$ but $x+s\in S$ for all $s\in S\setminus\{0\}$. We denote by $Pg(S)$ the set of pseudo-Frobenius numbers of S. The cardinal of $Pg(S)$ will be called the *type* of S and denoted by type(S). In [5], it is proved that a numerical semigroup is symmetric if and only if type$(S)=1$ (i.e. $Pg(S)=\{g(S)\}$).

The main result of this section is Theorem 15 which is an improvement of Theorem 4.

The following result is easy to prove.

Lemma 9. *Let S be a numerical semigroup generated by $\{n_1,\dots,n_p\}$ and let $x\in\mathbb{Z}$. Then x is a pseudo-Frobenius number of S if and only if $x\notin S$ and $x+n_i\in S$ for all $i\in\{1,\dots,p\}$.*

Using the previous lemma it is clear that if $S=\langle 5,6,7,8,9\rangle$, then $Pg(S)=\{1,2,3,4\}$. In general, if $S=\langle m,m+1,\dots,m+(m-1)\rangle$, then $Pg(S)=\{1,\dots,m-1\}$.

Given $n\in S\setminus\{0\}$ and $0=w(1)<w(2)<\cdots<w(n)$ the smallest elements of S in respective congruence classes mod n. We denote by Ap(S,n) (the *Apéry set* of n in S, called so after [1]) the set $\{0=w(1)<w(2)<\cdots<w(n)\}$. It is known (see [8]) that $w(n)=g(S)+n$ and S is symmetric if and only if $w(i)+w(n-i+1)=w(n)$ for all $i\in\{1,\dots,n\}$.

Let S be a numerical semigroup, we define in S the following partial order:

$$a\leqslant_S b \quad \text{if } b-a\in S.$$

From [5, Proposition 7] we deduce the following result.

Lemma 10. *If S is a numerical semigroup, $n \in S \setminus \{0\}$ and $\{w_{i1}, \ldots, w_{it}\} = \text{maximals}_{\leqslant_S} \text{Ap}(S, n)$, then $Pg(S) = \{w_{i1} - n, \ldots, w_{it} - n\}$.*

As a consequence of the previous result we obtain that $\text{type}(S) \leqslant m(S) - 1$.

A *MED-semigroup* is a numerical semigroup such that $m(S) = \mu(S)$. These semigroups have been widely studied (see for instance [2,9,12]). From Lemma 10 we get the following result.

Lemma 11. *Let S be a numerical semigroup. The following conditions are equivalent*:
(1) *S is a MED-semigroup*,
(2) *$\text{type}(S) = m(S) - 1$.*

Next we prove that the role that $Pg(S)$ plays in a numerical semigroup is analogous to the one played by $g(S)$ when the semigroup is symmetric.

Proposition 12. *Let S be a numerical semigroup, $g_1, \ldots, g_t$ be the pseudo-Frobenius numbers of S and $x \in \mathbb{Z}$. Then $x \notin S$ if and only if $g_i - x \in S$ for some $i \in \{1, \ldots, t\}$.*

Proof. If $x \notin S$ and $n \in S \setminus \{0\}$, then there exists $w \in \text{Ap}(S, n)$ and $k \in \mathbb{N} \setminus \{0\}$ such that $x = w - kn$. Let $\{w_{j1}, \ldots, w_{jt}\} = \text{maximals}_{\leqslant_S} \text{Ap}(S, n)$ and let $i \in \{1, \ldots, t\}$ be such that $w_{ji} - w \in S$. By Lemma 10 we can assume that $g_i = w_{ji} - n$. Then $g_i - x = w_{ji} - n - (w - kn) = (w_{ji} - w) + (k - 1)n \in S$.

Conversely, since $g_i - x \in S$ and $g_i \notin S$, we obtain that $x \notin S$. $\square$

Now we study sufficient conditions for a numerical semigroup to be an ISY-semigroup.

Proposition 13. *Let S be a numerical semigroup whose all pseudo-Frobenius numbers are odd. Then S is an ISY-semigroup.*

Proof. Suppose that $g_1, \ldots, g_t$ are the pseudo-Frobenius numbers of S. For each $i \in \{1, \ldots, t\}$ let S_{g_i} be a symmetric numerical semigroup such that $S \subseteq S_{g_i}$ and $g(S_{g_i}) = g_i$ (the existence of S_{g_i} follows by Lemma 2). We will see that $S = S_{g_1} \cap \cdots \cap S_{g_t}$. To this purpose, it is enough to prove that $S_{g_1} \cap \cdots \cap S_{g_t} \subseteq S$. Assume that $x \notin S$, then by Proposition 12, there exists $i \in \{1, \ldots, n\}$ such that $g_i - x \in S$ and thus $g_i - x \in S_{g_1} \cap \cdots \cap S_{g_t}$. Hence $g_i - x \in S_{g_i}$ and so $x \notin S_{g_i}$ (note that $g_i \notin S_{g_i}$). $\square$

The converse of Proposition 13 is not true in general, as the following example shows.

Example 14. Let $S = \langle 5, 21, 24, 28, 32 \rangle = \langle 5, 7 \rangle \cap \langle 5, 8 \rangle$ be an ISY-semigroup. Then $\text{Ap}(S, 5) = \{0, 21, 24, 28, 32\}$ and $\text{maximals}_{\leqslant_S} \text{Ap}(S, 5) = \{21, 24, 28, 32\}$. Using Lemma 10 we obtain that $Pg(S) = \{16, 19, 23, 27\}$. Note that S has an even pseudo-Frobenius number but it is an ISY-semigroup.

Theorem 15. *Let S be a numerical semigroup and $g_1 \ldots, g_t$ be its pseudo-Frobenius numbers. The following conditions are equivalent*:
(1) *S is an ISY-semigroup*,
(2) *for all g_i even, there exists an odd positive integer y_i such that $g_i + y_i \notin \langle S, y_i \rangle$.*

Proof. (1) $\Rightarrow$ (2) It is a consequence of Theorem 4.

(2) $\Rightarrow$ (1) If g_i is even, by Lemma 3, we deduce that there exists a symmetric semigroup S_{g_i} such that $S \subseteq S_{g_i}$ and $g_i \notin S_{g_i}$. The case g_i odd and the proof of $S = S_{g_1} \cap \cdots \cap S_{g_t}$ follows as in Proposition 13. □

As a consequence of the proof of (2) $\Rightarrow$ (1) of the previous theorem, we obtain the following result.

Corollary 16. *Let S be an ISY-semigroup with $type(S) = t$. Then S can be expressed as an intersection of t symmetric numerical semigroups.*

Now we describe an algorithmic method to express an ISY-semigroup as an intersection of symmetric numerical semigroups. From the proof of (2) $\Rightarrow$ (1) in Theorem 15 it suffices to determine, from a numerical semigroup with odd Frobenius number $g(S)$, a symmetric numerical semigroup $\bar{S}$ such that $S \subseteq \bar{S}$ and $g(\bar{S}) = g(S)$. To this purpose, the next result is crucial and has similar proof to the one of [8, Lemma 3.2] and it is also contained in the proof of [5, Proposition 4].

Lemma 17. *Let S be a non symmetric element of $\mathcal{S}(g)$ with $g(S) = g$ odd and set $h = \max\{x \in \mathbb{N} \mid x \notin S$ and $g - x \notin S\}$. Then $S \cup \{h\} \in \mathcal{S}(g)$.*

Let us consider the sequence of elements in $\mathcal{S}(g)$;
- $S_0 = S$,
- $S_{j+1} = S_j \cup \{h_j\}$, where $h_j = \max\{x \in \mathbb{N} \mid x \notin S_j$ and $g - x \notin S_j\}$.

Then, there exists $r \in \mathbb{N}$ verifying that $\{x \in \mathbb{N} \mid x \notin S_r$ and $g - x \notin S_r\} = \emptyset$. Clearly, S_r is a symmetric numerical semigroup such that $S \subseteq S_r$ and $g(S_r) = g$.

In order to illustrate this method, we give an example.

Example 18. Let $S = \langle 5, 21, 24, 28, 32 \rangle$ be a numerical semigroup. Then $g(S) = 27$. We compute a symmetric numerical semigroup $\bar{S}$ such that $S \subseteq \bar{S}$ and $g(\bar{S}) = 27$.

Note that $S = \{0, 5, 10, 15, 20, 21, 24, 25, 26\} \cup \{x \geqslant 28\}$
- $h_1 = \max\{x \in \mathbb{N} \mid x \notin S$ and $27 - x \notin S\} = 23$ and $S_1 = S \cup \{23\}$,
- $h_2 = \max\{x \in \mathbb{N} \mid x \notin S_1$ and $27 - x \notin S_1\} = 19$ and $S_2 = S \cup \{19, 23\}$,
- $h_3 = \max\{x \in \mathbb{N} \mid x \notin S_2$ and $27 - x \notin S_2\} = 18$ and $S_3 = S \cup \{18, 19, 23\}$,
- $h_4 = \max\{x \in \mathbb{N} \mid x \notin S_3$ and $27 - x \notin S_3\} = 16$ and $S_4 = S \cup \{16, 18, 19, 23\}$,
- $h_5 = \max\{x \in \mathbb{N} \mid x \notin S_4$ and $27 - x \notin S_4\} = 14$ and $S_5 = S \cup \{14, 16, 18, 19, 23\}$,
- $\{x \in \mathbb{N} \mid x \notin S_5$ and $27 - x \notin S_5\} = \emptyset$. Hence, $\bar{S} = S_5$ is a symmetric numerical semigroup generated by $\{5, 14, 16, 18\}$ with Frobenius number $g(S)$ containing S.

Now, we express $S = \langle 5, 21, 24, 28, 32 \rangle$ as an intersection of symmetric numerical semigroups.

Note that the pseudo-Frobenius numbers of S are $g_1 = 16$, $g_2 = 19$, $g_3 = 23$ and $g_4 = 27$ (see the example before Theorem 15). Note also that $16 + 7 \notin \langle S, 7 \rangle$ and therefore, by Theorem 15, we obtain that S is an ISY-semigroup. From the proof of (2) $\Rightarrow$ (1) in Theorem 15, we have that $S = S_{16} \cap S_{19} \cap S_{23} \cap S_{27}$:

- S_{16} is a symmetric numerical semigroup which contains $S' = \langle S, 7 \rangle \cup \{x \geqslant 24\}$ and $g(S_{16}) = g(S') = 23$.
- S_{19} is a symmetric numerical semigroup which contains $S' = S \cup \{x \geqslant 20\}$ and $g(S_{19}) = g(S') = 19$.
- S_{23} is a symmetric numerical semigroup which contains $S' = S \cup \{x \geqslant 24\}$ and $g(S_{23}) = g(S') = 23$.
- S_{27} is a symmetric numerical semigroup which contains $S' = S \cup \{x \geqslant 28\} = S$ and $g(S_{27}) = g(S') = g(S) = 27$.

Using the sequences described after Lemma 17 we obtain that $S_{16} = \langle 5, 7 \rangle$, $S_{19} = \langle 5, 11, 12, 13 \rangle$, $S_{23} = \langle 5, 12, 14, 16 \rangle$ and $S_{27} = \langle 5, 14, 16, 18 \rangle$.

Remark 19. Note that in the preceding example, S is expressed as an intersection of four symmetric numerical semigroups, though in Example 14 this same semigroup is expressed as an intersection of only two symmetric numerical semigroups. The algorithmic process described above does not supply the minimal decomposition of an ISY-semigroup.

4. Intersection of the symmetric numerical semigroups with the same Frobenius number

We say that a numerical semigroup is an *ISYG-semigroup* if $S = S_1 \cap \cdots \cap S_r$ where $S_1, \ldots, S_r$ are symmetric numerical semigroups such that $g(S_1) = \cdots = g(S_r) = g(S)$. In this section our aim is to study this kind of semigroups.

Lemma 20. *Let S be a numerical semigroup with odd Frobenius number g. The following conditions are equivalent*:
(1) *S is an ISYG-semigroup*,
(2) *for every $x \in \mathbb{Z} \setminus S$, we have that $g \notin \langle S, g - x \rangle$.*

Proof. (1) $\Rightarrow$ (2) Take $x \notin S$. Since S is an ISYG-semigroup, there exists a symmetric numerical semigroup $\bar{S}$ such that $S \subseteq \bar{S}$, $g(\bar{S}) = g$ and $x \notin \bar{S}$. Thus $g - x \in \bar{S}$ and therefore $g \notin \langle S, g - x \rangle$, because $\langle S, g - x \rangle \subseteq \bar{S}$ and $g \notin \bar{S}$.

(2) $\Rightarrow$ (1) For $x \in \mathbb{N} \setminus S$, let S_x be a maximal numerical semigroup containing $\langle S, g - x \rangle$ with $g(S_x) = g$. By Lemma 1, we know that S_x is a symmetric numerical semigroup with Frobenius number g and that $x \notin S_x$. It follows that $S = \bigcap_{x \in \mathbb{N} \setminus S} S_x$. $\square$

Lemma 21. *Let S be a numerical semigroup with $Pg(S)=\{g_1,\dots,g_t\}$ and $g(S)=g$. Then the following conditions are equivalent*:
(1) *for every $x\in\mathbb{Z}\setminus S$, we have that $g\notin\langle S,g-x\rangle$*,
(2) *$g\notin\langle S,g-g_i\rangle$ for all $i\in\{1,\dots,t\}$*.

Proof. (1) $\Rightarrow$ (2) It is trivial, because $g_i\in\mathbb{Z}\setminus S$ *for all* $i\in\{1,\dots,t\}$.

(2) $\Rightarrow$ (1) If $x\in\mathbb{Z}\setminus S$, then by Proposition 12, we know that there exists $i\in\{1,\dots,t\}$ such that $g_i-x\in S$. Assume that $s\in S$ is such that $g_i=x+s$. Then, since $g\notin\langle S,g-g_i\rangle=\langle S,g-x-s\rangle\supseteq\langle S,g-x\rangle$, we have that $g\notin\langle S,g-x\rangle$. □

As a consequence of Lemmas 20 and 21, we get the following result.

Theorem 22. *Let S be a numerical semigroup with odd Frobenius number g and $Pg(S)=\{g_1,\dots,g_t\}$. The following conditions are equivalent*:
(1) *S is an ISYG-semigroup*,
(2) *$g\notin\langle S,g-g_i\rangle$ for all $i\in\{1,\dots,t\}$*.

Assume that S is an ISYG-semigroup and hence it verifies condition (2) of the previous theorem. We denote by S_{g_i} a symmetric numerical semigroup with Frobenius number g such that $\langle S,g-g_i\rangle\subseteq S_{g_i}$. The existence of S_{g_i} follows by Lemma 1 and furthermore we can construct S_{g_i} using the procedure given after Lemma 17. Then $S=S_{g_1}\cap\cdots\cap S_{g_t}$. In fact, if $x\in\mathbb{Z}\setminus S$, then by Proposition 12, we know that $g_i-x\in S$ for some $i\in\{1,\dots,t\}$. Hence $g_i-x\in S_{g_i}$, since $S\subseteq S_{g_i}$. Then we can conclude that $g-x\in S_{g_i}$ and thus $x\notin S_{g_i}$.

Note that if $g\notin\langle S,g-g_{i1},\dots,g-g_{ik}\rangle$ with $\{i_1,\dots,i_k\}\subseteq\{1,\dots,t\}$, then we can take $S_{g_{i1}}=S_{g_{i2}}=\cdots=S_{g_{ik}}$.

Assume that $t\geqslant 2$ and $g_t=g$ (recall that $g\in Pg(S)$). Then using the previous remark we can take $S_{g_t}=S_{g_1}$ and thus $S=S_{g_1}\cap\cdots\cap S_{g_{t-1}}$, whence we have the following result.

Corollary 23. *Let S be a non symmetric ISYG-semigroup. Then $type(S)=t\geqslant 3$ and S can be expressed as an intersection of almost $t-1$ symmetric numerical semigroups with Frobenius number $g(S)$.*

In [5, Theorem 11] the following result is given.

Lemma 24. *If S is a numerical semigroup with $\mu(S)=3$, then $type(S)\in\{1,2\}$.*

As a consequence of Corollary 23 and Lemma 24, we have the following.

Corollary 25. *Let S be a numerical semigroup with $\mu(S)=3$. Then the following conditions are equivalent*:
(1) *S is an ISYG-semigroup*,
(2) *S is a symmetric numerical semigroup.*

Example 26. We prove that $S=\langle 6,11,15,20,25\rangle$ is an ISYG-semigroup. Moreover, applying the remark after Theorem 22, we see that S can be expressed as an intersection of symmetric numerical semigroups with the Frobenius number equal to 19. Note that

$$\mathrm{Ap}(S,6)=\{0,11,15,20,22,25\}$$

and $g(S)=19$. Then $\mathrm{maximals}_{\leqslant_S}\mathrm{Ap}(S,6)=\{15,20,22,25\}$, by Lemma 10, we have that $Pg(S)=\{g_1=9,\ g_2=14,\ g_3=16,\ g_4=g(S)=19\}$ and therefore $g(S)-g_1=10$, $g(S)-g_2=5$, $g(S)-g_3=3$ and $g(S)-g_4=0$. It is clear that $19\notin\langle S,10\rangle$, $19\notin\langle S,5\rangle$, $19\notin\langle S,3\rangle$ and $19\notin\langle S,0\rangle$. Hence, from Theorem 22, we deduce that S is an ISYG-semigroup.

Note that $19\notin\langle S,10,5,0\rangle$, whence we can take $S_{g_1}=S_{g_2}=S_{g_4}$ and this semigroup is symmetric with the Frobenius number $g(S_1)=19$ containing $\langle S,10,5,0\rangle$. Applying the method given after Lemma 17, we have that $S_{g_1}=S_{g_2}=S_{g_4}=\langle 5,6\rangle$.

Note also that S_{g_3} is a symmetric semigroup with $g(S_{g_3})=19$ and such that $\langle S,3\rangle\subseteq S_{g_3}$. Applying again the previous method we have that $S_{g_3}=\langle 3,11\rangle$.

Finally, $S=S_{g_1}\cap S_{g_2}\cap S_{g_3}\cap S_{g_4}=\langle 5,6\rangle\cap\langle 3,11\rangle$.

5. Numerical semigroups of type 2 that are intersection of symmetric numerical semigroups

Our goal in this section is to prove Theorem 29 which states that the converse of Proposition 13 is true for numerical semigroups of type 2.

Lemma 27. *Let S be a numerical semigroup, $x,y\in\mathbb{Z}$ and $s\in S$. If $x+y\notin\langle S,y\rangle$ and $x+y+s\notin S$, then $x+y+s\notin\langle S,y+s\rangle$.*

Proof. If $x+y+s\in\langle S,y+s\rangle$, then there exist $s'\in S$ and $a\in\mathbb{N}$ such that $x+y+s=s'+a(y+s)$. Since $x+y+s\notin S$, $a\neq 0$. Hence $x+y=s'+(a-1)s+ay\in\langle S,y\rangle$ a contradiction. □

Lemma 28. *Let S be a numerical semigroup, with pseudo-Frobenius numbers $g_1,\ldots,g_t$ and let $y\in\mathbb{Z}$ be such that $g_i+y\notin\langle S,y\rangle$ for some $i\in\{1,\ldots,t\}$. Then there exists $g_j\geqslant g_i+y$ such that $g_j\notin\langle S,g_j-g_i\rangle$.*

Proof. Since $g_i+y\notin S$, then by Proposition 12, we deduce that there exists g_j such that $g_j-(g_i+y)\in S$ and so $g_j=g_i+y+s$ for some $s\in S$. Hence, we have that $g_i+y\notin\langle S,y\rangle$, $g_i+y+s\notin S$ and $s\in S$. Using the previous lemma, we obtain that $g_i+y+s\notin\langle S,y+s\rangle$ and therefore $g_j\notin\langle S,g_j-g_i\rangle$ □

Now we can prove the following result.

Theorem 29. *Let S be a numerical semigroup with $type(S)=2$ and $Pg(S)=\{g_1<g_2\}$. The following conditions are equivalent*:
(1) *S is an ISY-semigroup,*
(2) *g_1 and g_2 are odd.*

Proof. (2) $\Rightarrow$ (1) Follows from Proposition 13.

(1) $\Rightarrow$ (2) By Lemma 6 we know that $g_2=g(S)$ is odd. If g_1 is even, then from Theorem 15, we deduce that there exists an odd number y such that $g_1+y\notin\langle S,y\rangle$. Using Lemma 28 we obtain that $g_2\notin\langle S,g_2-g_1\rangle$. Note that S satisfies condition (2) of Theorem 22 and so S is an ISYG-semigroup, which contradicts Corollary 23. $\square$

Example 30. Using Theorem 29, we deduce that $S=\langle 5,6,7\rangle$ is not ISY-semigroup because $Pg(S)=\{8,9\}$. Applying again Theorem 29 we have that $S=\langle 5,6,8\rangle$ is an ISY-semigroup since $Pg(S)=\{7,9\}$.

6. Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups with the same multiplicity

We say that a numerical semigroup S is an *ISYM-semigroup* if $S=S_1\cap\cdots\cap S_r$, with $S_1,\ldots,S_r$ symmetric numerical semigroups such that $m(S_1)=\cdots=m(S_r)=m(S)$ (remember that $m(S)$ denotes the multiplicity of S).

In this section, we suppose that S is a numerical semigroup with $m(S)\geqslant 3$. Note that if $m(S)=1$, then $\mathbb{N}=S$ and if $m(S)=2$ then $S=\langle 2,g(S)+2\rangle$; in both cases the semigroup S is symmetric.

Lemma 31. *Let S be a symmetric numerical semigroup with $m(S)\geqslant 3$. Then $g(S)\geqslant 2m(S)-1$.*

Proof. Note that $g(S)$ is odd and so $g(S)\geqslant 3$. If $g(S)<2m(S)-1$, then there exists $x,y\in\{1,\ldots,m(S)-1\}$ such that $x+y=g(S)$. Applying that S is symmetric, we deduce that $x\in S$ or $y\in S$, contradicting that $m(S)=\min S\setminus\{0\}$. $\square$

Lemma 32. *If m is an integer greater than or equal to 3, then $S=\langle m,m+1,\ldots,m+(m-2)\rangle$ is the unique symmetric numerical semigroup with $m(S)=m$ and $g(S)=2m-1$.*

Proof. By definition, it is obvious that S is symmetric. Now we need to show that S is unique. Suppose that $\bar{S}$ is a symmetric semigroup with $m(\bar{S})=m$ and $g(\bar{S})=2m-1$. We have that $\{1,\ldots,m-1\}\cap\bar{S}=\emptyset$, therefore,

$$\{(2m-1)-1,\ldots,(2m-1)-(m-1)\}\subseteq\bar{S}$$

and thus $m,m+1,\ldots,m+(m-2)\in\bar{S}$. Hence we conclude that $S=\bar{S}$. $\square$

Lemma 33. *Let S be a numerical semigroup such that $m(S) \geqslant 3$ and $g(S)$ is odd. The following conditions are equivalent*:
(1) $g(S) \geqslant 2m(S) - 1$,
(2) *there exists a symmetric numerical semigroup $\bar{S}$ such that $S \subseteq \bar{S}$ and $m(\bar{S}) = m(S)$.*

Proof. (1) $\Rightarrow$ (2) Let $\bar{S}$ be the symmetric numerical semigroup obtained from S by using the recurrent method exposed after Lemma 17. Now, it is enough to see that $m(S) = m(\bar{S})$. In fact,

$$h_j = \max\{x \in \mathbb{N} \mid x \notin S_j \text{ and } g(S) - x \notin S_j\} > \frac{g(S)}{2} \geqslant \frac{2m(S) - 1}{2}$$

and so $h_j \geqslant m(S)$.

(2) $\Rightarrow$ (1) Follows from Lemma 31 (note that $g(S) \geqslant g(\bar{S})$). □

Using the previous results we can characterize the ISYM-semigroups.

Theorem 34. *Let S be a numerical semigroup with $m(S) \geqslant 3$, $g(S)$ odd and $g(S) \geqslant 2m(S) + 1$. The following conditions are equivalent*:
(1) *S is an ISYM-semigroup*,
(2) *for every $x \in \mathbb{N} \setminus S$ with $x > g(S)/2$, there exists $y \in N$ such that*
 (i) $x + y \geqslant 2m(S) - 1$,
 (ii) *$x + y$ is odd*,
 (iii) $x + y \notin \langle S, y \rangle$,
 (iv) *if $y \neq 0$, then $y \geqslant m(S)$.*

Proof. (1) $\Rightarrow$ (2) Since S is an ISYM-semigroup, for $x \in \mathbb{N} \setminus S$, there exists a symmetric numerical semigroup $\bar{S}$ such that $S \subseteq \bar{S}$, $m(S) = m(\bar{S})$ and $x \notin \bar{S}$. If we choose $y = g(\bar{S}) - x$, then, since $\bar{S}$ is symmetric, $g(\bar{S})$ is odd and $y \in \bar{S}$. Hence (ii) and (iv) are satisfied. Furthermore, by Lemma 31, (i) is fulfilled. Finally, (iii) is verified too, since $x + y = g(\bar{S}) \notin \bar{S} \supseteq \langle S, y \rangle$.

(2) $\Rightarrow$ (1) Let $\bar{S}$ be a symmetric numerical semigroup obtained from S by using the recurrent method exposed after Lemma 17. In the proof of Lemma 33 we saw that $m(\bar{S}) = m(S)$. It is clear that if $x \in \bar{S} \setminus S$, then $x \notin S$, $g(S) - x \notin S$ and $x > g(S)/2$. We will see that there exists a symmetric numerical semigroup S_x such that $S \subseteq S_x$, $m(S_x) = m(S)$ and $x \notin S_x$. Let $y \in \mathbb{N}$ verifying (i)–(iv) and set $S' = \langle S, y \rangle \cup \{x + y + 1, x + y + 2, \ldots\}$. Then S' is a numerical semigroup with multiplicity $m(S)$ and the Frobenius number $x + y \geqslant 2m(S) - 1$. Take S_x a symmetric numerical semigroup such that $S' \subseteq S_x$, $m(S_x) = m(S)$ and $g(S_x) = x + y$ (the proof of (1) $\Rightarrow$ (2) in Lemma 33 guarantees the existence of S_x). Furthermore, $x \notin S_x$, since $y \in S_x$ and $g(S_x) = x + y$. Clearly, $S = \bar{S} \cap (\bigcap_{x \in \bar{S} \setminus S} S_x)$ and therefore S is an ISYM-semigroup. □

Finally we illustrate the previous results with some examples.

Example 35. $S = \langle 5, 6, 8, 9 \rangle$ is a numerical semigroup with $m(S) = 5$ and $g(S) = 7$. As $g(S) < 2m(S) - 1$, then by Lemma 33 we obtain that S is not an ISYM-semigroup.

Example 36. $S=\langle 6,11,15,20,25\rangle$ is a numerical semigroup with $m(S)=6$ and $g(S)=19$. Taking $x=16$, we have that $16\in\mathbb{N}\setminus S$, $19-6=3\notin S$ and $16>\frac{19}{2}$. It is clear that the unique natural number y such that $x+y$ is odd and $x+y\notin\langle S,y\rangle$ is $y=3$. Hence S is not an ISYM-semigroup because the condition (iv) of Theorem 34 is not satisfied for $y=3$.

Example 37. $S=\langle 5,21,24,28,32\rangle$ is a numerical semigroup with $m(S)=5$ and $g(S)=27$. It is easy to see that $\{x\geqslant 14\mid x\notin S \text{ and } 27-x\notin S\}=\{14,16,18,19,23\}$. Taking $x=14$ we can use $y=5$ which verifies conditions (i)–(iv) of Theorem 33. Analogously, for $x=16$, we can use $y=7$, for $x=18$ we can use $y=5$ and for $x=23$, we can use $y=0$. Hence, by Theorem 34 we deduce that S is an ISYM-semigroup. Note that S can be expressed as an intersection of symmetric numerical semigroups with multiplicity 5, for this we apply the method that it is deduced from the proof of (2) $\Rightarrow$ (1) in Theorem 34.

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